Falling Rate of Profit Under Constant Rate of Exploitation

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Abstract

The Okishio Theorem is often used to nullify Marx's law of tendency of the rate of profit to fall, which, however, relies on a peculiar assumption: constant real wage. This paper attempts to re-evaluate Marx's law by instead using his original assumption: constant rate of exploitation. It turns out that if a capital-using and labor-saving technical change takes place in the consumption goods department, the general rate of profit will *definitely* fall; if it takes place in the capital goods department, whether the rate of profit will rise or fall depends on the initial state of the economy as well as the relative magnitude of the technical change, but a meaningful threshold condition governing its direction of movement is derived.

Keywords: Falling rate of profit; technical change; constant rate of exploitation; Okishio Theorem

JEL classification: B51; C67; D51

1 Introduction

In February 2019, Marx's tomb in Highgate cemetery in London was vandalized first by a hammer, and then red paints. (Guardian 2019a,b) Such physical attacks on Marx's tomb were very uncommon before; the intellectual controversies on Marx's legacies, however, have been incessant. A few years ago, one of the MEGA② editors, Heinrich (2013a), refueled the debate on Marx's law of tendency of the rate of profit to fall (henceforth, *LTRPF*, or *Marx's law*, or *the law*, or *the main tendency*): By drawing on textual evidence from MEGA②, he conjectures that Marx had once

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striven, but failed to prove, and eventually might have abandoned his own theory of falling rate of profit. This, of course, stimulated some intense exchanges (Kliman et al. 2013; Carchedi and Roberts 2013; Mage 2013; Moseley 2013; Heinrich 2013b). More lately, Kurz (2018), with a similar reading of MEGA②, accuses Engels of editing away Marx's own uncertainty about his law. Both contributions represent an exegetical approach to revisit Marx's law, though an obituary of the law has long been written (Van Parijs 1980).

Whatever Marx' own thought process once was, the central *scientific* question still remains: Is this law correct? Can it be proved using modern analytical techniques? To what degree?

This paper, inspired by Laibman (1982), offers a "semi-positive" answer to the above question in an input-output framework[1]: Under the assumption of constant rate of exploitation, if a capital-using and labor-saving (henceforth, *CU-LS*) technical change takes place in the consumption goods department (henceforth, *Dept-Con*), the general rate of profit will *definitely* fall; if it takes place in the capital goods department (henceforth, *Dept-Cap*), the direction of movement in the rate of profit will depend on the initial production technique and distribution of the economy as well as the relative magnitude of the technical change.

2 Marx's assumption

This paper exclusively examines the theoretical debate on Marx's LTRPF. A disclaimer is necessary here: We do not pursuit any argument regarding the net tendency, or the actual movement of the rate of profit; we only investigate what Marx called the main tendency, that is, the tendency of rate of profit to fall due to rising organic composition of capital (henceforth, *OCC*). Let's start with a review of Marx's original theory, and then Heinrich's critique which represents a critical exegetical approach. To establish LTRPF, Marx assumed an economy-wide uniform rate of exploitation. According to his definition of the rate of profit,[2]

$$r = \frac{S}{C+V} = \frac{S/V}{1+C/V} = \frac{e}{1+VCC} = \frac{e}{1+OCC \cdot \tilde{I}^C \cdot (1/\tilde{I}^b) \cdot (1/\tilde{w})} \tag{1}$$

[1]More specifically, the economy is set to consist of two departments and multiple sectors, under single-production with capital stock.

[2]Note that the movement of this rate of profit in value terms need not reflect that of the rate of profit in price terms. (Roemer 1981: 94-97)

where S, V, and C are respectively surplus value, variable capital, and constant capital at the aggregate level; r, e, and VCC stand for the general rate of profit, the rate of exploitation, and the value composition of capital; OCC is the composition of capital evaluated at constant values and constant real wage, $\tilde{I}^C$ is the index of the unit value of intermediate inputs, $\tilde{I}^b$ is the index of the unit value of consumption articles, $\tilde{w}$ is the index of real wage (Shaikh 1990; Saad-Filho 1993).[3] The rate of profit r would *tend* to fall, were the rate of exploitation e to be held constant, and the assumed rising OCC to dictate the movement of VCC. However, the change in OCC will necessarily *induce* change in unit values of commodities according to Equation (3) below. This induced change was treated by Marx as one of the counter-tendencies; nonetheless we think it is actually part of the main tendency since it is *necessarily* induced by the changing OCC. With this consideration, it is not obvious that a rising OCC will lead to a falling tendency in the rate of profit (Morishima 1973: 35). The argument that mechanization would create a downward trend in the rate of profit is questionable.

When it comes to the actual dynamics of the rate of profit, Marx was fully aware of other counter-tendencies that might offset the main tendency. It is debatable, but beyond the scope of this paper, whether the main tendency dominates all the counter-tendencies; Heinrich, however, contends that the main tendency alone cannot be substantiated, different from our above argument regarding the induced change in relative prices, in that he thinks it is impossible to maintain the constancy of the rate of exploitation in general, thus the main tendency must also include the effect of the rising rate of exploitation caused by the same technical change that raises the OCC. He further makes a case that Marx actually did try to incorporate such a rising rate of exploitation, but only to arrive at an incorrect proof because of his ignoring how fast the constant capital would increase compared to the decline in variable capital which raises the rate of exploitation. Using an counter-example, he then concludes that Marx's LTRPF is indeterminate. His indeterminacy argument rests on the questionable impossibility to maintain a constant rate of exploitation. The truth is, if we allowed the real wage to vary, it is completely plausible in general to maintain such a constancy so that the main tendency is formulated in the context of *class struggle neutrality* as

[3] "The value-composition of capital, inasmuch as it is determined by, and reflects, its technical composition, is called the organic composition of capital." (Marx 1962: 143) More specifically, "The variable capital thus serves here (as is always the case when *the wage is given*) as an index of the amount of labor set in motion by a definite total capital." (142) "On the other hand, a difference in the magnitude of the constant capital may *likewise* be an index of a change in the mass of means of production set in motion by a definite quantity of labor-power." (144)

is proposed by Laibman (1982: 103):

> [I] propose a *constant rate of exploitation* as the hallmark of class struggle neutrality. It is the rate of exploitation — the ratio of unpaid to paid labor time — which expresses the balance of class forces at a given time. It is when wages keep in step with productivity that the balance of class forces does not change; this then, is the neutral framework in which to analyze technical change.

Several influential commentators have come to either disprove or prove Marx's LTRPF in a rather different framework: constant real wage. At one extreme, Okishio (1961) proves that a cost-reducing technical change will raise the general rate of profit;[4] at the other, Morishima (1973) discovers that, after a neutral (the unit values of all commodities remain constant[5]), capital-using and labor-saving technical change, the general rate of profit will necessarily fall. Though under the same constant real wage assumption, in terms of cost effectiveness, the technical changes considered by Okishio and Morishima are just as diametrically exclusive to each other as their conclusions: a neutral technical change as defined by Morishima is not cost-reducing (Roemer 1981), which is also confirmed by the extended Okishio Theorem derived later in this paper.

It is without question that Morishima confines technical change to too narrow a subset: A technical change in the production process would generally alter the unit values of all commodities that are directly or indirectly required for production in the industry where technical change takes place. There isn't any mechanism in a capitalist economy that guarantees the existence of such a technical change, let alone its viability. Okishio (2001) himself actually discards the assumption of constant real wage four decades after the birth of his theorem, but in the first place did Marx ever make such an assumption? Many opponents of Marx's law ascribe a constant real wage assumption to Marx, for example, Kurz (2018: 21), and Roemer (1981: 145, 150-153), with the former just taking it for granted without much argument, and the latter wanting to explicitly impose a literal subsistence wage theory to Marx's theory on the value of labor power, and dispel the historical and social elements in wage determination.[6] As is also acknowledged by Roemer, textual evidence

[4]The Okishio Theorem is mathematically well established and robust with the considerations of fixed capital (Nakatani and Hagiwara 1978; Roemer 1979), joint production, differential turnover time(Roemer 1979), and product innovation (Nakatani and Hagiwara 1997).

[5]In this case, the rate of exploitation also remains constant.

[6]Bidard (2004: 74) is an excellent exception: "It is often written that the law has been refuted by Okishio (1961)...But this statement[the Okishio Theorem] is not an appropriate answer to Marx's problem since it assumes that the real wage is constant when production per worker increases and, therefore, the rate of exploitation increases. Nevertheless, the law should be discussed under the hypothesis of a constant rate of exploitation as initially conceived by Marx." Foley (2009: 138-9) is also well aware of the pervasive misreading of Marx's assumption.

suggests that Marx seldom had constant real wage in mind, and always assumed the value of labor power, hence the rate of exploitation, to be constant in his formulation of LTRPF.

In the opening paragraph in Chapter XIII, Volume III of *Capital* (Marx 1962: 207) where Marx officially laid out his law, Marx wrote:

> Assuming a given wage and working-day, a variable capital, for instance of 100, represents a certain number of employed labourers. It is the index of this number. Suppose £100 are the wages of 100 labourers, for, say, one week. If these labourers perform equal amounts of necessary and surplus labour, if they work daily as many hours for themselves, i.e., for the reproduction of their wage, as they do for the capitalists, i.e., for the production of surplus-value, then the value of their total product=£200, and the surplus-value they produce would amount to £100. The rate of surplus-value, $\frac{s}{v}$,would=100%.

It is clear from this paragraph that the assumption of "a given wage" is in value terms, namely, a given value of labor power, rather than a given real wage since it is improper to use the latter as the denominator to define the rate of surplus value. If we read on, his assumption becomes crystal clear:

> If it is further assumed that this gradual change in the composition of capital is not confined only to individual spheres of production, but that it occurs more or less in all, or at least in the key spheres of production, so that it involves changes in the average organic composition of the total capital of a certain society, then the gradual growth of constant capital in relation to variable capital must necessarily lead to *a gradual fall of the general rate of profit*, so long as the rate of surplus-value, or the intensity of exploitation of labour by capital, remain the same.

It is only by dispensing with the constant real wage assumption and adopting a class-struggle-neutral framework can LTRPF be fairly judged. Nonetheless, it should be pointed out that it is not out of the purpose of merely defending Marx that we defend his assumption; instead, we think a constant rate of exploitation setting is an appropriate, or neutral as perceived by Laibman, reference point of departure when we examine the effect of technical change on profitability, for the process of introducing new production technique is never void of class struggle.

3 Choice of technique

Within the analytical political economy literature, especially along the Neo-Ricardian tradition, capitalists are often the only protagonists in the choice of production technique. New production techniques are assumed to be exogenously given, and whether it is viable for production only depends on whether it is able to reduce the unit cost of production and generate a temporary super profit. Bidard (2004: 78)'s market algorithm in Figure 1 best illustrates this conventional perception of the choice of technique. Taking production technique and the real wage rate as given, the Neo-Ricardians then go on to criticize the Marxian labor theory of value by arguing for its irrelevance in calculating the price of production as well as the general rate of profit. Only do we demystify these unexplained givens can we address the Neo-Ricardian critique and see the true relevance of the labor theory of value.

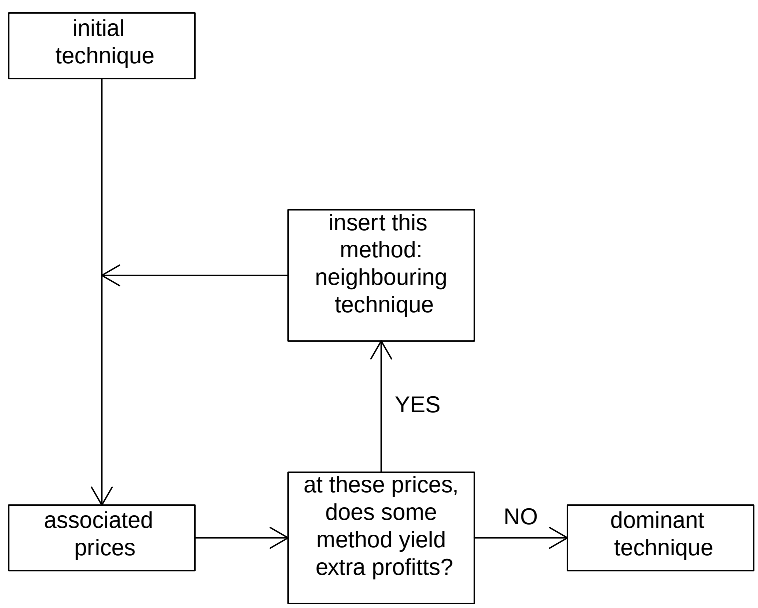

Figure 1: Neo-Ricardian choice of technique

Source: Bidard (2004: 78) .

The most important place to be demystified is the labor process that most economics, except the Marxian tradition, treat as a black box. In his response to the Neo-Ricardian critique, Shaikh hits the point that it is the performance of the labor process that determines the relation between inputs and outputs which characterizes a production technique. (1987: 296) There is rich historical and contemporary evidence that the new technique anticipated by the capitalist based on the current prices and wage must be adapted or even discarded, after it is introduced, according to class struggle on the shop floor. The most telling and forcible example is the failure of the introduction of the powerlooms in the late 18th century England:

> The first powerloom factory, established in 1792 with four hundred powerlooms run by steam, was totally destroyed by a hostile crowd of weavers. For many years thereafter the powerloom was rarely used; sixteen years later there were only about thirty powerloom factories, and all of them were small. One of the largest employers of hand weavers at that time, with four thousand employees under the putting out system, testified before a Parliamentary committee that there would have been ten thousand powerlooms at work within ten years of their first use had it not been for fear of having the looms destroyed. (Clawson 1980: 67-8)

The above failure was just one episode of the violent history of labor's riots against the introduction of machinery which brought about massive unemployment and lower wages at the dawn of the industrial revolution in England; it is so violent that England at that time had to pass a law to felonize the destruction of machines with death penalty. (Mantoux 1961: 399-401)

As is observed by Mantoux, class struggle against technical change "is not peculiar to any one period or country." (1961: 399) A decade into the 20th century, when Henry Ford put into use the first assembly lines for the production of his popular car Model T in 1913 in the United States, workers registered their dissatisfaction with the repetitive and tedious minute operations on the line simply by a "labor flight" out of his plants:

> The turnover of his working force had run, he was to write, to 380 percent for the year 1913 alone. So great was labor's distaste for the new machine system that toward the close of 1913 every time the company wanted to add 100 men to its factory personnel, it was necessary to hire 963. (Sward 1968: 32; as cited in Braverman 1974)

So Ford's famous "five dollars a day" offer, as opposed to the previous 2.34 dollar a day, was not purely out of his benevolence to improve the poor living conditions of his workers, but instead a natural response to the rebellion of his labor force: to lure workers back to the unpleasant assembly line work. Rather than being violent as in the powerloom story, this rebellion took the peaceful form of quitting, which facilitated by the high labor demand at that time. (Braverman 1974: 146-151)

The fact that Ford had to double the wage rate in order to keep running his assembly lines, which soon to become an iconic production technique for the era of mass production, casts doubt on the often-made assumption of constant real wage regarding the choice of technique. To introduce a new production technique, capitalists have to secure the compliance of their workers who are no passive agents, using carrot or stick. Workers will certainly react to the introduction of a new production technique in one way or another; sometimes they are even able to limit capitalists' set of new techniques before they are introduced. In the early 1970s, unions in Europe was faced with the wave of introducing the then-new computer-based information and control systems in production. Some of them responded to this challenge in a rather active way. For example, Iron and Metal Workers Union in Norway was among the first to create a "data shop steward" position to learn and demystify the new technology. "The task of the data shop steward, and the union in general, is to engage, as effectively as possible, in a struggle over information and control, a struggle engaged in, with equal sophistication and earnestness, by the other side."(Noble 1979: 49) Equipped with sufficient knowledge of the new technology, these unions were usually successful in making the capitalist side to remove all the monitoring and controlling functions of the new technology that are detrimental to labor's interests.(Noble 1979: 47-9) This active practice to "contest management control over new technology" was soon learned by the American unions "such as the tool and die workers in the UAW Local 600 at the Ford River Rouge plant, and the 'new technology committee' members of IUE Local 201 at the General Electric Lynn River Works."(Noble 2017: 349-50)

The above historical examples are just about labor on the defense in capitalists' choice of technique. Now, most intriguingly, China's IT workers has recently demonstrated the potential of labor to take the offensive with advanced technology. In the past two decades, there has been a pervasive practice among many IT companies in China to impose a "9-9-6" work schedule (workday: 9 am-9 pm; 6 days a week) on their programmers, which is a blatant violation of the Chinese labor law. Most recently, more and more IT companies are pushing openly for this work schedule especially combined with layoffs and a reduction in workers' compensation due to the current slowdown of capital accumulation in China. Concomitant with this practice has been outcries of programmers in the cyberspace as well as some scattered media coverage, especially when there is death from overwork. Discontents and angers accumulate, though most of the time labor is not organized. In March 2019, Chinese programmers eventually began to unite themselves nationwide and started a movement (project) called “996.ICU” on the GibHub website which serves as a hub for programmers to share their work. Within a few weeks, this project soon became the most starred (popular) one ever on that site. “The name 996.ICU refers to ‘Work by ‘996’, sick in ICU’, an ironic saying among Chinese programmers, which means that by following the ‘996’ work schedule, you are risking yourself getting into the ICU (Intensive Care Unit).”(Github 2019; Abacus 2019) Programmers from all over the country not only report companies and their practices that violate the Chinese labor law, but have also developed a software license that is meant to prohibit these companies from using any open-source programs that embed this license, until they have rectified their illegal practice. The effect of this movement has not yet fully played out, but censorship of the GitHub website by some major Chinese browsers is already in place. (Vice 2019) No matter to what extent it will succeed, the “996.ICU” movement has amazingly shown the world that the laboring class have the great potential to wield the cutting edge of advanced technology against capital.

These episodes, from different times and places, have revealed a fact that under capitalist mode of production, whenever capitalists want to introduce a technical change that is cost-reducing, as long as it involves the re-organization of labor in the production process either by lengthening the working day to produced absolute surplus value or by substituting machines for labor to produce relative surplus value , workers will respond, most often in negative ways, if not always aggressive, because of the tension created by the inherent tendency for the capitalists to intensify labor and reduce workers’ control over the labor process (Clawson 1980: 27).

Therefore, with class struggle absent, the Neo-Ricardian “rosy” theory of choice of technique is a far cry from being an adequate representation of the actual process of technical change. A complete theory should necessarily consider class struggle in the labor process, and a class-struggle-neutral framework as proposed by Laibman can serve as a proper starting point in examining the effect of technical change on profitability, while the actual shift of relative class power is subject to historical contingencies.

There are several attempts to examine the theory of falling rate of profit in such a class-struggle-neutral framework along two lines. The first line adopts a constant wage-profit ratio assumption as by Franke (1999), and Roemer (1978); the second line adopts a constant rate of exploitation assumption as by Bidard (2004), and Laibman (1982).

Constant wage-profit ratio can be seen as a close candidate to constant rate of exploitation in terms of class struggle neutrality. Franke is able to show that, if the two classes maintain a constant share of the net output after a fixed-capital-using and labor-saving, cost-reducing technical change, the general rate of profit will fall. However, such an interesting result heavily relies on assuming no change in the circulating capital. Since fixed capital is quantitatively equal to circulating capital multiplied by its corresponding turnover time (Brody 1970: 35-41), the underlying turnover time in Franke's setting is prolonged and plays a key role in driving the general rate of profit down. If circulating capital were to also increase, as he admits, the movement in the rate of profit will become ambiguous. Therefore, the effect of mechanization is entangled with the effect of lengthening turnover time which might not be a real trend in capitalist development.

Roemer assumes the wage-profit ratios to be different across sectors but remain unchanged after the technical change, so as to avoid the consideration of output levels. In his simplistic two-sector model, he demonstrates that a capital-using and labor-saving technical change, if happening in the capital good sector, will lead to a fall in the general rate of profit; if it happens in the consumption good sector, the general rate of profit will remain the same. Nonetheless, his model admits an arguably inconsistency in terms of the level of abstraction: The rate of profit is assumed to be equalized while the real wage rates and the wage-profit ratios are different across sectors. This inconsistent treatment is essential in arriving at his result.

Bidard is among the very few opponents who recognize Marx's real assumption. In spite of this admittance, he too conveniently cancels Marx's law with an over-simplified one-good model which captures the indeterminacy in the law. His indeterminacy argument itself is nothing wrong, as we will verify later, but more interesting results could have been obtained if one used a more disaggregated model like Laibman does. Laibman finds that, in a two-sector economy, if the rate of exploitation is constant, a capital-using and labor-saving technical change in the consumption sector will reduce the general rate of profit; but if such a technical change happens in the capital good sector, the general rate of profit will either rise or fall. Is his insight from this simplistic two-sector model also generally true? This paper, among other things, provides an affirmative answer with a two-department and multiple-sector model, which is also generalized to take into account the presence of capital stock in the Appendix C.

4 The input-output model

Suppose in a closed economy where there are m different sectors (processes), each producing a single commodity. Assume turnover times of all inputs to be unity so that no capital stock is involved (However, under the more general case considering capital stock, our main theorem still holds, which is proved in Appendix C). Without loss of generality, let sector 1 through n produce pure capital goods that are only used as intermediate inputs in the production processes and not consumed by workers; sector $n+1$ through m produce pure consumption goods only consumed by workers. Notations and definitions of variables are as follows:

- $A = (a_{ij})$, $m \times m$, intermediate input coefficient matrix, where $a_{ij} \geq 0$ denotes the amount of commodity i required to produce 1 unit of commodity j.
- $L = (l_1, l_2, \cdots, l_m)$, $1 \times m$, direct labor input vector, where $l_i > 0$ denotes the amount of labor required to produce 1 unit of commodity i.
- $b = (b_{n+1}, b_{n+2}, \cdots, b_m)^T$, $(m-n) \times 1$, a reference consumption basket, or basic basket, where $b_i \geq 0$ denotes the amount of commodity i that appears in this basket.[7]
- $\bar{b} = (0, \cdots, 0, b^T)^T$, $m \times 1$, augmented from b with 0's in the first n elements.
- w, scalar, index of real wage, or the real wage rate. For tractability of the model, we assume the real wage to be equal to $wb = (wb_{n+1}, wb_{n+2}, \cdots, wb_m)'$; that is, when the real wage is changing, only the size of workers consumption bundle is changing, while the proportions among different consumption goods remain the same.
- $p = (p_1, p_2, \cdots, p_m)$, $1 \times m$, price vector, where $p_i \geq 0$ denotes the price of commodity i.
- $x = (x_1, x_2, \cdots, x_m)^T$, $m \times 1$, output vector, where $x_i \geq 0$ denotes the output level of commodity i.
- $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$, $1 \times m$, value vector, where $\lambda_i \geq 0$ denotes the labor value per unit of commodity i.

[7]Throughout this paper, superscript T means transposition.

In this economy, sectoral rates of profits are assumed to equalize to π, so that the price and the value systems are respectively characterized by

$$p = (1+\pi)(pA + pw\bar{b}L) = (1+\pi)pM \tag{2}$$

$$\Lambda = \Lambda A + L \tag{3}$$

where $M \equiv A + w\bar{b}L$. Assume A is *productive* and M is *reproducible*,[8] namely in a Marxian equilibrium, each industry is earning a positive equal rate of profit $\pi > 0$. And the value vector is solved at $\Lambda = L(I-A)^{-1} > 0$. Further assume A is *indecomposable*, thus is M. According to the Perron-Frobenius theorems, there exists a unique solution to M's positive eigenvalue, $1/(1+\pi)$, associated with a positive eigenvector. Namely, $\pi > 0$ exists and is unique for a price vector $p > 0$ which is also unique up to multiplication by scalars. (Nikaido 1970: 107-154) In addition, for mathematical convenience, we use a numeraire condition to derive absolute prices: $p\bar{b} = 1$.[9] In this framework, we can immediately derive some useful propositions.

Proposition 1 (Wage-profit frontier)**.** *For a given production technique (A, L) and a given basic basket b, the real wage rate w is negatively related with the general rate of profit π, i.e., the wage-profit frontier is downward sloping.*

Proof. We know for given (A, L) and b, M is increasing in w. By the Perron-Frobenius theorems, M's largest eigenvalue $1/(1+\pi)$ is increasing in M. Therefore, π is decreasing in w. □

This proposition, though standard in mathematical economics, mirrors the antagonism between the capitalist and working classes, for under any given production technique, capitalists earning more always implies workers earning less, and vice versa. Next, as a reference point, let's reproduce the Okishio Theorem and develop an extended version of it. Note that, throughout this paper, without loss of generality (Roemer 1981: 213), we only consider technical change in one sector at a time.

[8]Namely $\exists x \in R^m_+$ such that $x \geq Ax$, and $x \geq Mx$. Throughout this paper, $A \geq B \Leftrightarrow A \geqq B$ *but* $A \neq B$.

[9]Note that the choice of numeraire will not affect our calculation or interpretation of the distributional variables which are only affected by the relative prices determined without the numeraire. The numeraire chosen here is just for mathematical convenience; a more sensible one in Marxian terms should be the value preservation axiom (TE1) as in Appendix A: "total value equals total price," which, if adopted, would only complicate our technical aspect without adding much analytical power regarding the purpose of this paper.

Definition 1 (Cost-reducing technical change)**.** *A technical change in sector j, $(A_j, l_j) \mapsto (A'_j, l'_j)$, is cost-reducing if and only if it lowers the unit cost of production under the wage and prices before the technical change, i.e.,*

$$pA_j + pw\bar{b}l_j > pA'_j + pw\bar{b}l'_j \tag{4}$$

where A_j refers to the jth column of A.

This is also called the *viability condition* in the literature, for a rational capitalist will only adopt a technical change that is cost-reducing.[10] It is indeed only a necessary condition for the viability of the new production technique, because the cost-reducing technical change has to pass the test of class struggle in the labor process as argued above: it won't be viable if the workers fight strong enough against the new production technique. Besides, we can define a non-cost-reducing technical change simply as a negation of the above definition: A technical change is non-cost-reducing if and only if it doesn't lower the unit cost of production under the wage and prices before the technical change, i.e., $pA_j + pw\bar{b}l_j \leq pA'_j + pw\bar{b}l'_j$. With cost effectiveness defined, we have the following proposition:

Proposition 2. *If the rate of profit is fixed, a cost-reducing technical change will increase the real wage.*

Proof. If the rate of profit were to be fixed after a technical change, the new price system is

$$p' = (1 + \pi)(p'A' + w'L') \tag{5}$$

and relative prices are normalized using $p'\bar{b} = 1$. Let $\lambda = 1/(1+\pi)$, then from the two price systems we have $p = wL(\lambda I - A)^{-1}$ and $p' = w'L'(\lambda I - A')^{-1}$. Further more, since $p\bar{b} = p'\bar{b} = 1$,

$$wL(\lambda I - A)^{-1}\bar{b} = w'L'(\lambda I - A')^{-1}\bar{b} \tag{6}$$

Now, for a cost-reducing technical change, we have the following procession of inequalities, each implied by the its preceding step: $pA + wL \geq pA' + wL'$, $p(A' - A) \leq w(L - L')$, $wL(\lambda I - A)^{-1}[(\lambda I -$

[10] There is a debate on whether the cost-reducing condition is *the* criterion governing capitalists' choice of technique (Shaikh 1978, 2016; Roemer 1979; Nakatani 1980). However, the main conclusions of this paper don't rely on where we stand in this debate.

$A) - (\lambda I - A')] \leq w(L - L')$, $L - L(\lambda I - A)^{-1}(\lambda I - A') \leq L - L'$, $L(\lambda I - A)^{-1} \geq L'(\lambda I - A')^{-1}$. Note that, according to the Perron-Frobenius theorems, both $(\lambda I - A)^{-1}$ and $(\lambda I - A')^{-1}$ exist and are non-negative because the Perron-Frobenius roots have the following relationships: $1 > \lambda = \lambda(M) > \lambda(A)$ and $1 > \lambda = \lambda(M') > \lambda(A')$, since $M > A$ and $M' > A'$ where $M' \equiv A' + w'\bar{b}L'$, by the indispensability of direct labor inputs. The last step together with Equation (6) implies $w' > w$. □

Next, the Okishio Theorem can be easily inferred as a corollary to Proposition 1 and 2:

Proposition 3 (The Okishio Theorem). *If the real wage is fixed, a cost-reducing technical change will increase the general rate of profit.*

Proof. By Proposition 2, a cost-reducing technical change will increase the real wage rate when the rate of profit is fixed. Then by the post-technical-change downward sloping wage-profit frontier (Proposition 1), to reduce the real wage rate to its initial level, the rate of profit has to increase. □

These three propositions together reveal a fact that a cost-reducing technical change essentially increases the economic efficiency of production. The fruits of this increase could be appropriated either by the capitalist class or the working class, or shared between the two. Since the Okishio Theorem only presents the sufficiency of the technical change being cost-reducing for a rise in the rate of profit, next we establish the necessity using the following lemma:

Lemma 1. *If the real wage is fixed, a non-cost-reducing technical change will decrease or keep constant the general rate of profit.*

Proof. The proof is only different from that of the Proposition 3 by the direction of a sign. First reuse the proof of Proposition 2 up to Equation (6). Now for a non-cost-reducing technical change, we have a similar procession of inequalities with all signs flipped to the other direction: $pA + wL \geqq pA' + wL'$, $p(A' - A) \geqq w(L - L')$, $wL(\lambda I - A)^{-1}[(\lambda I - A) - (\lambda I - A')] \geqq w(L - L')$, $L - L(\lambda I - A)^{-1}(\lambda I - A') \geqq L - L'$, $L(\lambda I - A)^{-1} \leqq L'(\lambda I - A')^{-1}$. The last step together with Equation (6) implies $w' \leqq w$. Then by the post-technical change downward sloping wage-profit frontier, to maintain the real wage rate to its initial level, the rate of profit has to decrease or remain constant(when the inequality is binding). □

Now we are able to derive a more complete version of the Okishio Theorem:

Theorem 1 (Extended Okishio Theorem)**.** *Assume the real wage is fixed, the general rate of profit will increase if and only if the technical change is cost-reducing.*

Proof. The sufficiency of this theorem is proved by the Okishio Theorem; necessity is proved by contradiction: When the general rate of profit increases, suppose the technical change is non-cost-reducing, by Lemma 1 the rate of profit shouldn't have increased, which is a contradiction. So the technical change must be cost-reducing. □

This theorem, by extending the Okishio Theorem, establishes a complete correspondence between the movement of the rate of profit and the cost effectiveness of a technical change under the assumption of constant real wage. Next, let's turn to Marx's law.

5 Evaluating Marx's law

First of all, it is necessary to point out the distinction between the value rate and price rate of profit. Marx defined the general rate of profit as total surplus value divided by total capital advanced in value terms; it is called the value rate of profit in the literature,

$$r = \frac{(1 - \Lambda w\bar{b})Lx}{\Lambda Ax + \Lambda w\bar{b}Lx} = \frac{e}{VCC(x) + 1} \tag{7}$$

where, for given output x, the value composition of capital, $VCC(x) = (\Lambda Ax)/(\Lambda w\bar{b}Lx)$ and rate of surplus value, $e = (1 - \Lambda w\bar{b})/(\Lambda w\bar{b})$. Marx's formula is thus re-established in this framework. This value rate of profit is, however, in general not equal to the price rate of profit (Roemer 1981: 90-97), which is exactly why the transformation problem exists under this dual value-price system. One could clearly see this difference by noting that the price system is compatible with any growth theory, in that any variable including the price rate of profit π in the price system will remained unaltered whatever the output vector x is, while the value rate of profit r is a function of output x, among other things. Nonetheless, the transformation problem could dissolve if the output x is in the possibility set of solutions to the transformation problem (see Appendix A).

Given the peculiarity of the possibility set, thus the transformation problem, we will proceed with the price rate of profit: On the one hand, it is outside the scope of this paper to develop a

Marxian growth theory; on the other, the value rate of profit is about the generation of surplus value, regardless of how it is realized in different industries, while the price rate of profit is about how much surplus value that individual capitalists can capture with their capital advanced, and therefore is of their utmost direct concern[11] But we will retain the rate of exploitation variable which measures the ratio of surplus labor time to necessary labor time or the relative class power. It is a central variable to any class society, failing to recognize which will certainly lead one to the illusion that technological progress is the ultimate force that liberates all human beings.[12]

We follow the tradition of distinguishing between Dept-Cap and Dept-Con (The luxury goods sectors are assumed away). Recall that sector 1 through n are the capital goods sectors and the rest, sector $n+1$ through m, are the consumption goods sectors. Then accordingly, $a_{ij} = 0$, for $i = n+1, \cdots, m$ and $j = 1, \cdots, m$. The technical data become $A = \begin{pmatrix} A_I & A_{II} \\ 0 & 0 \end{pmatrix}$ and $L = (L_I, L_{II})$, in which A_I and A_{II} denote the intermediate inputs coefficient matrices of Department I (capital goods) and II (consumption goods); L_I and L_{II} are their respective direct labor input vectors. All other related matrices and vectors are partitioned accordingly. The price and value systems then become

$$p_I = (1+\pi)(p_I A_I + p_{II} wbL_I) \tag{8}$$

$$p_{II} = (1+\pi)(p_I A_{II} + p_{II} wbL_{II}) \tag{9}$$

$$\Lambda_I = \Lambda_I A_I + L_I \tag{10}$$

$$\Lambda_{II} = \Lambda_I A_{II} + L_{II} \tag{11}$$

and the numeraire condition becomes $p_{II} b = 1$. Under the two-department setting, the rate of exploitation becomes $e = (1 - \Lambda_{II} wb)/(\Lambda_{II} wb) = 1/(\Lambda_{II} wb) - 1$.

[11]Marx also acknowledged this point in Volume III of *Capital*; that's why he had to deal with the transformation of surplus values into profits.

[12]Imagining the economic prospect 100 years ahead from 1930, Keynes (2010) once wrote, "Thus for the first time since his creation man will be faced with his real, his permanent problem — how to use his freedom from pressing economic cares, how to occupy the leisure, which science [and technology] and compound interest will have won for him, to live wisely and agreeably and well." "Three-hour shifts or a fifteen-hour week may put off the problem for a great while. For three hours a day is quite enough to satisfy the old Adam in most of us!" Now, almost 100 years after Keynes' prediction, the standard working day is still 8 hours in the developed countries, which itself is a result of the labor movement, and workers in the developing countries like China are working 12 hours a day, either on an aforementioned "9-9-6" schedule or by a "voluntary" over-time schedule (Pun and Chan 2012).

Now the question is, holding relative class power constant, will a capital-using and labor-saving technical change that reflects the process of mechanization decrease the price rate of profit? Let's first formally define the technical change we concern here,[13]

Definition 2 (Capital-using and labor-saving technical change)**.** *A technical change in sector j, $(A_j, l_j) \mapsto (A'_j, l'_j)$, is capital-using and labor-saving (CU-LS) if and only if the production process uses more intermediate inputs and less direct labor input evaluated at the prices and wage before technical change, i.e.,*

$$pA_j < pA'_j \quad \textit{and} \quad wl_j > wl'_j \tag{12}$$

In the Morishima-Roemer tradition, capital-using is often defined as at least one intermediate input in a production process increases and all else remain constant, which is a rather limiting definition in that it rules out the substitution among different types of intermediate inputs. Our definition here overcomes that limitation by allowing all intermediate inputs to change in either direction, as long as their summation, aggregated by the same weights (in this case, constant prices and wage) increases. It is also more in the spirit of Marx's concept of OCC which is essentially value composition of capital evaluated at constant values as mentioned above: A CU-LS technical change as defined here would mean a rising (price-analogue of) OCC. Moreover, our definition is more economically reasonable and more consistent with the definition of a cost-reducing technical change, for when capitalists consider changing the production technique, it is the expenses of material and labor inputs, rather than the their physical quantities, that they are trading off; it is the prices and wage before technical change, rather than future prices that they are faced with.

So what's the relationship of a cost-reducing and a CU-LS technical change? The short answer is there will always be an overlapping between them. Figure 2 illustrates such intersection. For any production technique $\{A, L\}$ and their associated prices and wage rate $\{p, w\}$, there is a "cost-space" under the old price and wage and the new production technique. The point (wl, pA) on the downward-sloping iso-cost line corresponds to the original production technique. Any production technique corresponding to the cost structure below the iso-cost line is cost reducing; if above, non-cost-reducing. Within the cost-reducing region, any production technique above the original intermediate input cost will be CU-LS. Intuitively, for any CU-LS technical change, if the labor

[13]Similar definition has been made by Flaschel, Franke, and Veneziani (2013).

cost that is saved is more than enough to cover the increases in the intermediate input cost, then it is cost-reducing.

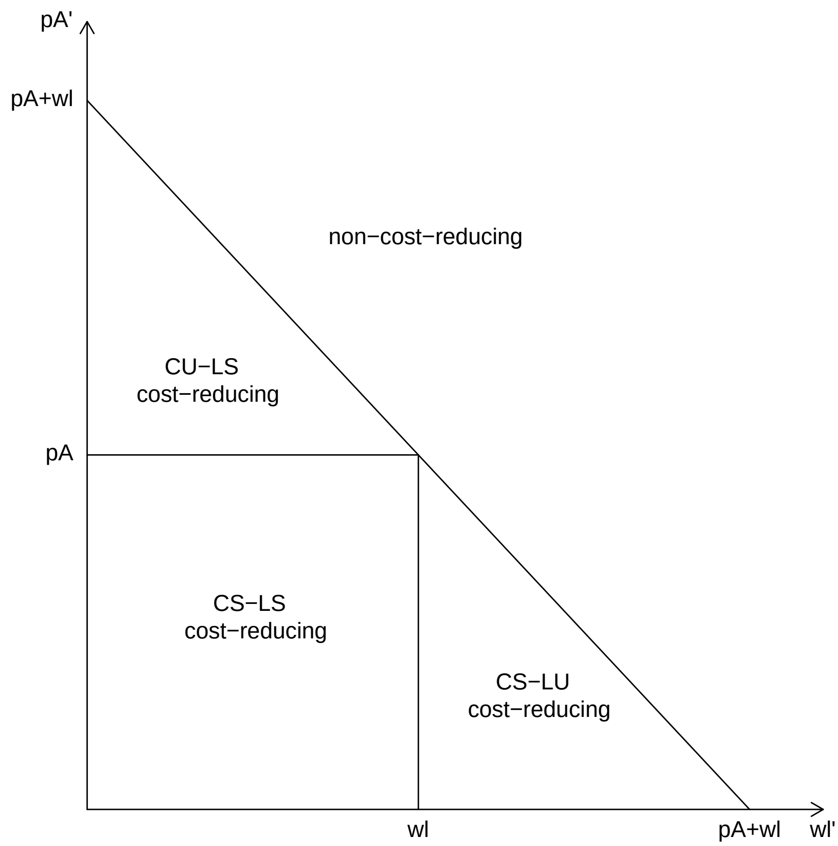

Figure 2: Types of technical change

Note: Without explicitly given their formal definitions (which can be readily inferred from the definition of CU-LS), CS-LS means the capital saving and labor saving, and CS-LU means capital saving and labor using.

With the above definitions and clarifications, the following theorem can be proved as our answer to the central question of this paper.

Theorem 2. *Assuming the rate of exploitation to be constant, for a CU-LS technical change,*

(i) If it takes place in Dept-Con, then the price rate of profit will fall.

(ii) If it takes place in Dept-Cap, then the direction of movement of the rate of profit will be indeterminate, depending on the initial state of the economy as well as the relative magnitude of the technical change.

Some remarks first: This theorem applies to any CU-LS technical change in Dept-Con, including the cost-reducing ones which always exist, as shown above. The strong result of part (i) is thus exempt from the debate regarding whether cost-reducing is *the* criterion by which a capitalist adopts a new technique, for in our results, as long as the technical change is CU-LS, part (i) holds.

Proof. (i) Only a sketch of the proof and intuitions of this part will be provided here; for the complete technicalities of the proof, see Appendix B.

Step 1. Let the real wage index w adjust to w' such that the rate of profit remains unchanged after a CU-LS technical change in Dept-Con, same as what we do in the proof of Proposition 2 but under a two-department setting. It turns out:

$$\frac{w' - w}{w} = \frac{[w(L_{II} - L'_{II}) - p_I(A'_{II} - A_{II})]b}{[p_I A'_{II} + wL'_{II}]b} \tag{13}$$

What's the intuition behind this growth rate? Further assume the technical change is cost-reducing and recall that $p_{II}b = 1$, then the square bracket in the numerator represents the amount of cost saved per unit of output in Dept-Con, and then the numerator is the cost saved per basic basket under pre-technical-change prices. Likewise, the denominator represents the cost per basic basket under pre-technical-change prices but new production technique. Therefore the right hand side of the equation means the cost saving rate per basic basket. If the real wage were to increase just enough to offset the cheapening of the consumption goods, all the benefits resulting from the technical change will be completely grabbed by workers and the rate of profit earned by the capitalists remains the same.

Step 2. Now assume the rate of exploitation remains constant before and after the technical change, thus both the new real wage rate and the new rate of profit will presumably be different from the old ones. Denote the new levels as w^* and π^*. By the constancy of the rate of exploitation, the value of labor power should remain constant, i.e., $\Lambda_{II}wb = \Lambda'_{II}w^*b$, with which we have

$$\frac{w^* - w}{w} = \frac{\Lambda_{II}b - \Lambda'_{II}b}{\Lambda'_{II}b} = \frac{[w(L_{II} - L'_{II}) - w\Lambda_I(A'_{II} - A_{II})]b}{[w\Lambda_I A'_{II} + wL'_{II}]b} \tag{14}$$

Intuitively, the right hand side just means the rate of reduction in the value of the basic basket. If the real wage grew at that same rate, the value of labor power will remain constant, and so will the rate of exploitation.

Step 3. Now compare the two growth rates in the real wage. It's obvious that the difference between w' and w^* lies in the difference between p_I and $w\Lambda_I$. By the Perron-Frobenius theorems, we can prove $p_I > w\Lambda_I$. Therefore, $w^* > w'$. Further by Proposition 1, we have $\pi^* < \pi$.

(ii) The following example suffices to prove this part.

Table 1: Different CU-LS and cost-reducing technical changes in Dept-Cap

Case 0	Case 1	Case 2
$A = \begin{pmatrix} 0.13 & 0.11 \\ 0 & 0 \end{pmatrix}$	$A' = \begin{pmatrix} 0.16 & 0.11 \\ 0 & 0 \end{pmatrix}$	$A' = \begin{pmatrix} 0.22 & 0.11 \\ 0 & 0 \end{pmatrix}$
$L = (0.017, 0.013)$	$L' = (0.013, 0.013)$	$L' = (0.012, 0.013)$
$e = 1$	$e = 1$	$e = 1$
$b = 0.13$	$b = 0.13$	$b = 0.13$
$w = 242.7$	$w^* = 248.9$	$w^* = 250.0$
$\pi = 75.6\%$	$\pi^* = 77.2\%$	$\pi^* = 75.0\%$

Case 0 is the initial state of the economy. From Case 0 to Case 1 we have a CU-LS and cost-reducing technical change in Dept-Cap: a 20% increase in capital input, and a 20% reduction in direct labor input. The result is, while the real wage increases to maintain the constancy of rate of exploitation, rate of profit also rises. From Case 0 to Case 2, only the magnitude of the technical change is different: a 68% increase in capital input and a 28% reduction in direct labor input, which is still cost-reducing. The effect is, real wage rises more than that in Case 1 and the rate of profit falls below its previous level. □

It's worth making some further comments about this theorem. First of all, this theorem also holds in a more general model with capital stock (see Appendix C). Secondly, regarding the proof of part (i) of the theorem, it is astonishing that it actually does not require any stringent restriction on the cost-effectiveness of the technical change: Any CU-LS technical change happening in Dept-Con will definitely lead to a fall in the general rate of profit, as long as the rate of exploitation is held constant. Marx's law is partially re-established here.

Figure 3 illustrates the comparative statics of rates of profit under different assumptions for a CU-LS technical change in Dept-Con. The horizontal axis denotes the initial level of the real wage rate w; with given production technique $\{A, L\}$, the black line represents the wage-profit frontier

$\pi(A, L, b, w)$ that we start out with; the blue line represents the wage-profit frontier under Okishio's constant real wage assumption after the technical change, call that $\pi^{OK}(A', L', b, w)$; the red line represents the wage-profit frontier under Marx constant rate of exploitation assumption for given initial wage level after the technical change, call that $\pi^{MA}(A', L', b, w^*(w))$. To compute the red line, we starts with a initial wage level w, then we compute $w^*(w)$ according to Equation (14) which keeps constant the rate of exploitation, finally, we use w^* to look for the corresponding rate of profit in the blue line that corresponds to the new production technique. In this example, it is easily seen that the red line lies entirely under the original wage-profit frontier.

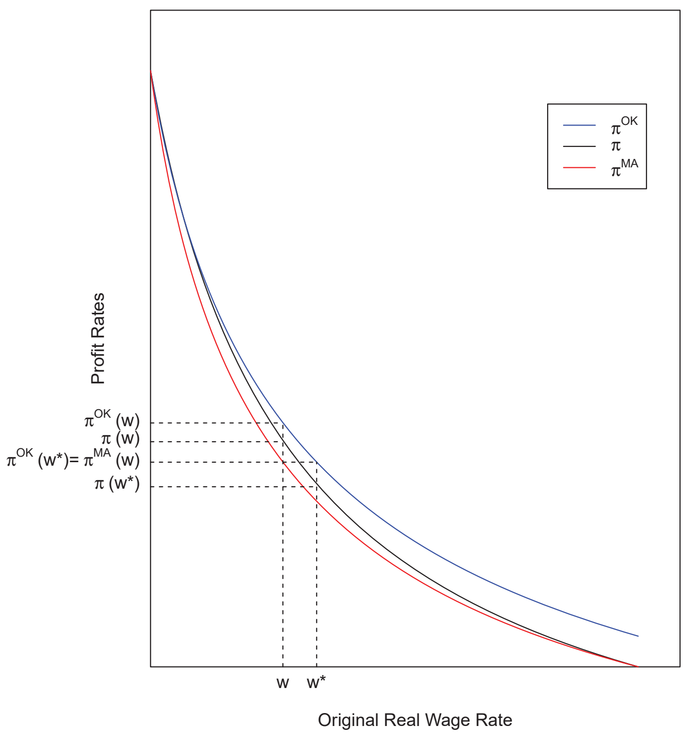

Figure 3: Profit rates before and after a CU-LS technical change in Dept-Con

Note: This graph is generated using simulated data.

Further more, it could be proved that for a CU-LS and cost-reducing technical change, both wage growth rates are positive: The real wage will have to grow to maintain the constancy of the rate of profit (Proposition 2) or rate of exploitation because a CU-LS and cost-reducing technical change is progressive in that the value vector will decrease (Roemer 1981: 102-3). Secondly, the indeterminacy in part (ii) of the theorem does not substantiate Marx's law in that only some CU-LS technical changes under certain circumstances will lead to a fall in the rate of profit. Let there be a CU-LS technical change in Dept-Cap, similar to the procedure as outlined in the proof of part (i), we can derive the two real wage growth rates for Dep-Cap:

$$\frac{w' - w}{w} = \frac{(p_I - p'_I)A_{II}b}{wL_{II}b} \tag{15}$$

$$\frac{w^* - w}{w} = \frac{(\Lambda_I - \Lambda'_I)A_{II}b}{(\Lambda'_I A_{II} + L_{II})b} \tag{16}$$

The numerator in the right hand side of Equation (15) means the cost saved per basic basket because of the cheapening of intermediate inputs, and the denominator means labor cost per basic basket. If the real wage grows at this cost "saving" rate, all the costs saved in the intermediate input costs from the technical change will be reallocated to higher labor costs that supports a higher real wage, the capitalists retain the same rate of profit. The interpretation of Equation (16) is the same as in part (i).

We can now see w' and w^* have similar structures, and their relative magnitude depends not only on the relative scale (illustrated by Table 1) of the technical change, but also on the initial production techniques as well as initial distribution. Figure 4 illustrates the last dependent factor. Marx's line intercepts with the original wage-profit frontier at a certain initial real wage rate. If we start out with a real wage rate to its left, this given technical change will result in a decrease in the rate of profit under the assumption of constant rate of exploitation; if to its right, the rate of profit will rise. Notice that the Marx-original intersection takes place to the right of the Okishio-original intersection, where cost-reducing scenarios are represented (Proposition 1), there will be some cost-reducing technical change that results in a fall in the rate of profit were the rate of exploitation to be held constant. Now comes a question: If the rate of profit had fallen due to constant rate of exploitation, under the new set of prices and wage, will capitalists have an incentive to reverse back to their original production technique? The answer depends on the shape of the new wage-profit frontier after the technical change. If it takes a "normal" shape as in Figure

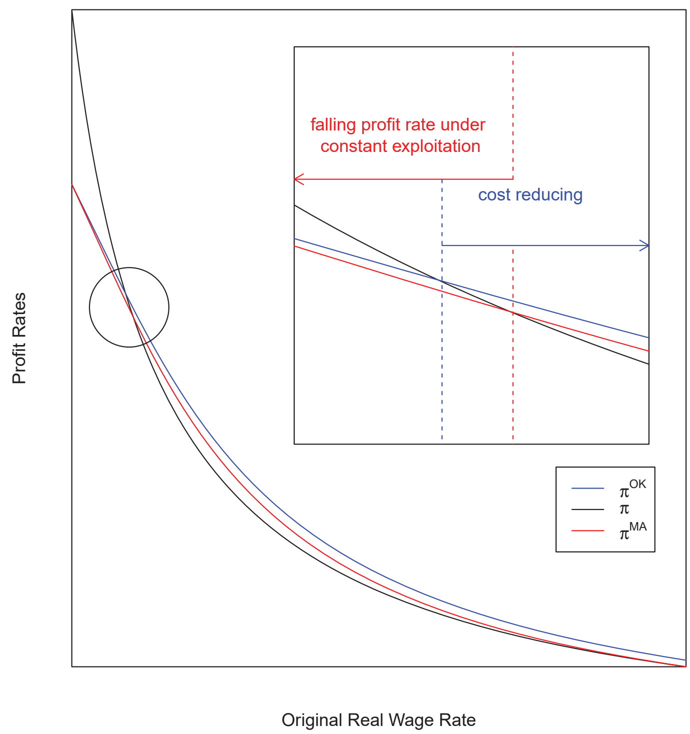

Figure 4: Profit rates before and after a CU-LS technical change in Dept-Cap

Note: This graph is generated using simulated data .

4, after the technical change, the capitalists face a wage rate of w^*, at which the new technique permits a higher rate of profit than the general rate of profit that the old technique would do, namely $\pi^{OK}(w^*) > \pi(w^*)$; then according to the extended Okishio Theorem, reversing back from the new technique to the old is not cost reducing. However, if the new wage-profit frontier π^{OK} takes an "abnormal" shape as in Figure 5, which is possible according to the Sraffian "re-switching" literature, switching back to the old technique is cost-reducing given w^*, since $\pi^{OK}(w^*) < \pi(w^*)$. Empirically, the possibility of these abnormalities is quite low, for example, only 3.65% out of 4389 tested cases for 9 industrialized OECD countries between 1968 and 1990 (Han and Schefold 2006), which doesn't dampen the relevance of our results by much.

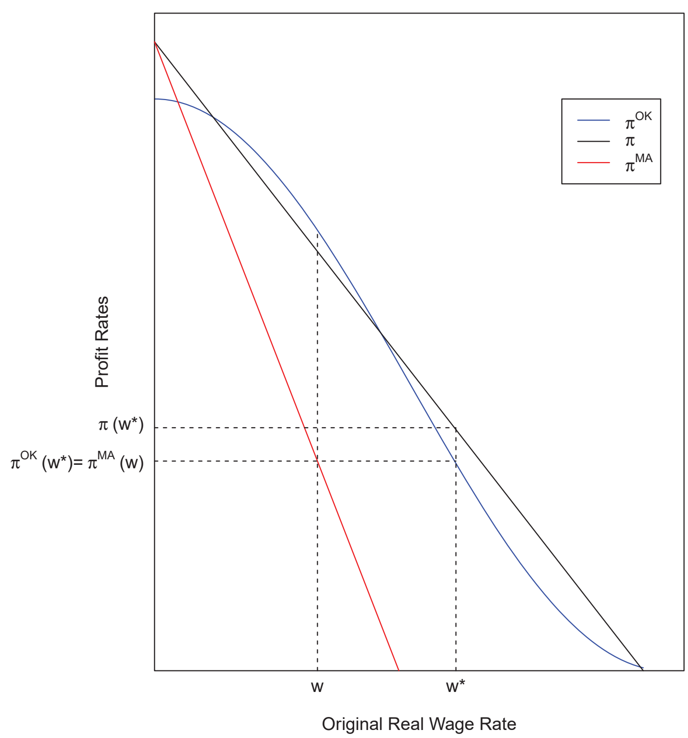

Figure 5: Re-switching of production technique in Dept-Con

Note: For the ease of illustration, this graph is not generated using simulated data

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Finally, to wrap up the discussion, the following proposition highlights the threshold of real wage growth governing the rise and fall of the price rate of profit, which is a generalized version of Foley (2009: 137-9)'s and Basu (2018)'s "Marx-Okishio threshold condition."

Proposition 4 (Threshold Condition)**.** *After a technical change, if the real wage grows at a rate*

$$\begin{cases} \textit{greater than} \\ \textit{equal to} \\ \textit{smaller than} \end{cases} \hat{w}', \textit{ then the rate of profit will } \begin{cases} \textit{fall} \\ \textit{remain constant} \\ \textit{rise} \end{cases}$$

where $\hat{w}' = (w' - w)/w$ *which is the cost-saving ratio per basic basket as in Equation (13) or (15).*

Proof. The middle case is true by the construction of $\hat{w}'$, and the other two cases are implied by the downward sloping wage-profit frontier (Proposition 1). □

6 Concluding Remarks

It has been proved that, in the above input-output model, Marx's law of tendency of the rate of profit to fall, with the price rate of profit being considered, is half correct and half indeterminate, in that under the assumption of constant rate of exploitation (in the spirit of Marx), if a capital-using and labor-saving technical change (mechanization) takes place in the consumption goods department, the general rate of profit will *definitely* fall; if it takes place in the capital goods department, the direction of movement of the general rate of profit depends on the initial state of the economy as well as the relative magnitude of the technical change: it could rise, or fall, or remain constant. This result proves the merits of sector-level analysis since the effect of technical change is predicated on in what type of sector it happens. Given the indeterminacy in the law, the net effect of technical changes in different sectors at the same time will be ambiguous. It is because of this ambiguity that empirical research is invited to locate where the indeterminacy falls in reality. The above conclusion is also valid in a more general model with the presence of capital stock that includes both fixed capital and inventories.

Moreover, by revealing the essence of the Okishio Theorem and using historical examples, we have highlighted the important role of class struggle in the process of technical change and argued for a class-struggle-neutral framework as a proper reference point to study the distributional effects of mechanization (its actual effects will be subject to historical contingencies).

Finally, the model and insights presented in this paper might be directly relevant for the analysis of the dynamics of rate of profit in the regulated phase of capitalism where the rate of exploitation remains relatively stable; while the Okishio theorem might be relevant for the neo-liberal phase where we witness the stagnation of real wages and a rising rate of exploitation (Mohun 2005).

Appendices

A On the transformation problem

The value rate defined by Marx, for given output x, is

$$r = \frac{(1 - \Lambda w\bar{b})Lx}{\Lambda Ax + \Lambda w\bar{b}Lx} = \frac{(1 - \Lambda w\bar{b})Lx/\Lambda w\bar{b}Lx}{\Lambda Ax/\Lambda w\bar{b}Lx + 1} = \frac{e}{VCC(x) + 1} \tag{17}$$

where the value composition of capital,

$$VCC(x) = \frac{\Lambda Ax}{\Lambda w\bar{b}Lx} \tag{18}$$

and rate of surplus value,

$$e = \frac{1 - \Lambda w\bar{b}}{\Lambda w\bar{b}} = \frac{1}{\Lambda w\bar{b}} - 1 \tag{19}$$

The two total equalities that Marx was trying to restore after the transformation process are

(I) Total value equals total price of production

$$\Lambda x = px \tag{TE1}$$

(II) Total surplus value equals total profit

$$r(\Lambda Ax + \Lambda w\bar{b}Lx) = \pi(pAx + pw\bar{b}Lx) \tag{TE2}$$

In the input-output framework, the first thing to note is any variable including the price rate of profit π in the price system will remained unaltered whatever the output vector x is while the value rate of profit r is a function of output x, among other things. The next thing to note is the price system has 2 degrees of freedom, because we have m equations and $m+2$ unknowns p, π, and w. However, according to the Perron-Frobenius Theorems, to determine distribution factors π and w we only need to reduce 1 degree of freedom, by taking π or w as given, which will also determine relative prices, namely the proportions among the elements of p. Let's take w as given. To finally determine absolute prices and the whole price system, we then can use (TE1) as normalization for any non-negative output x. The problem is, we have used up both degrees of freedom and there is no guarantee that (TE2) will hold in general for any x.

It is well known that, in a dual value-price system, if the economy is on the von Neumann balanced growth path, the transformation problem dissolves because the value rate of profit will be equal to the price rate of profit, so that the two total equalities can hold simultaneously Morishima (1973). However peculiar it might be, the possibility set of solutions to the transformation problem is non-empty; what's more important, it's non-empty under the general circumstances where OCC in different processes could be arbitrarily different.

However, is the von Neumann balanced growth path the only element to this possibility set under such settings? This question hasn't been explored, at least in the literature. The answer is negative. The crucial condition for the two total equalities to hold at the same time is the equality between the value rate of profit and price rate of profit. In general, they are not equal.[14]

Here comes the idea of the possibility set of solutions to the transformation problem which is a set of such outputs that the price rate of profit equal the value rate of profit, namely

$$\mathcal{X} \equiv \{x \in R^m_+ | r(x; A, w, b, l) = \pi(A, w, b, l)\} \tag{20}$$

Now let's look at the properties of $\mathcal{X}$. With $\{A, w, b, l\}$ given, π and e are uniquely determined. Then the condition $r = \pi$ is equivalent to $VCC(x) = e/\pi - 1 = (e-\pi)/\pi$, by (17). The ratio $(e-\pi)/\pi$, for want of a better name, let's call it the conversion rate from the rate of exploitation into the rate of profit, and denote it as t. Then further by (18), we have

$$\Lambda A x = t \Lambda w \bar{b} L x \tag{21}$$

[14]For details on the relationship between value and price rates of profit, see (Roemer 1981: 90-97)

which can be rearranged into

$$\Lambda(A - tw\bar{b}l)x = 0 \tag{22}$$

Therefore,

$$\mathcal{X} = \{x \in R^m_+ | \Lambda(A - tw\bar{b}l)x = 0\} \tag{23}$$

We can see $\mathcal{X}$ is an open set bounded from below lying on the hyper-plane perpendicular to the vector $\Lambda(A - tw\bar{b}l)$.

The possibility set of solutions is *ad hoc* by its construction. So is it useful to solve the transformation problem? Yes, if one were to approach the transformation problem from the point of output. Any growth theory designed to solve the transformation problem has to find its path lying in the possibility set. In this sense, the possibility set narrows down our search for a meaningful growth theory to the transformation problem. Further exploration along this line is left for the future. But we have to admit that in the very general, the transformation problem cannot be solved in this value-price dual system.

So, does this insolubility matter for the Marxian labor theory of value? Not really. The crux of the Marxian labor theory of value lies at the claim that all values in the economic systems originate from productive labor (TE1), which is compatible with the price system in general by serving as its numeraire condition (only with this numeraire is the prices in this input-output framework the Marxian prices of production which is transformed labor values). Therefore we can retain the first total equality safely. What about total surplus values equals total profits, which corresponds to the argument that surplus value is the source of profit? We can, unfortunately, admit it is a result of imprecision of Marx's analysis due to the lack of sufficient analytical tools in his time, of which he is well aware (Marx 1962: 162). Why don't we dispense with TE2 and instead claim that the source of profit is *distorted* surplus value (Laibman 2018), now that the Fundamental Marxian Theorem (Morishima 1973; Bowles and Gintis 1978) ensures profits qualitatively originate from surplus values: When value is transformed into prices of production, surplus value will not exactly transform into profits, just as cost of production in value terms will not exactly transform into cost of production in price terms; instead, total values will be spread, without leakages, into all components of the price system through the profit rate equalization process, with no guarantee of an exact translation in its components.

B Proof of Theorem 2 (i)

Theorem 2 (i) states: Assuming the rate of exploitation constant, for a CU-LS technical change, if it takes place in Dept-Con, then the price rate of profit will fall.

Proof. This part is proved in 3 steps.

Step 1. Let the real wage index w adjust so as to maintain constancy of the rate of profit after a CU-LS technical change in Dept-Con. The price system before the technical change is

$$p_I = (1+\pi)(p_I A_I + p_{II} w b L_I) \quad or \quad p_I = w L_I [1/(1+\pi) I - A_I]^{-1} \tag{24}$$

$$p_{II} = (1+\pi)(p_I A_{II} + p_{II} w b L_{II}) \tag{25}$$

$$p_{II} b = 1 \quad (numeraire\ equation) \tag{26}$$

The price system after the technical change is

$$p'_I = (1+\pi)(p'_I A_I + p'_{II} w' b L_I) \quad or \quad p'_I = w' L_I [1/(1+\pi) I - A_I]^{-1} \tag{27}$$

$$p'_{II} = (1+\pi)(p'_I A'_{II} + p'_{II} w' b L'_{II}) \tag{28}$$

$$p'_{II} b = 1 \quad (numeraire\ equation) \tag{29}$$

Changes in the prices of Dept-Cap is calculated as

$$p'_I - p_I = (w' - w) L_I [1/(1+\pi) I - A_I]^{-1} \tag{30}$$

$$= \frac{(w' - w)}{w} p_I \tag{31}$$

Prices in Dept-Con will change according to

$$\begin{aligned} p'_{II} - p_{II} &= (1+\pi)(p'_I A'_{II} - p_I A_{II} + w' L'_{II} - w L_{II}) \\ &= (1+\pi)[(p'_I - p_I) A'_{II} + p_I (A'_{II} \\ &- A_{II}) + (w' - w) L'_{II} + w (L'_{II} - L_{II})] \end{aligned} \tag{32}$$

Post multiply the price differences of Department II by b and apply the numeraire equations, we have

$$\begin{aligned} 0 &= (p'_{II} - p_{II})b \\ &= (1+\pi)[(p'_I - p_I)A'_{II} + p_I(A'_{II} - A_{II}) \\ &+ (w' - w)L'_{II} + w(L'_{II} - L_{II})]b \end{aligned} \tag{33}$$

Since $\pi > 0$ by the reproducibility of M, $1 + \pi > 0$; and we have

$$[(p'_I - p_I)A'_{II} + p_I(A'_{II} - A_{II}) + (w' - w)L'_{II} + w(L'_{II} - L_{II})]b = 0 \tag{34}$$

Rearrange the four terms into two parts and use Equation (31), then we have

$$(w' - w)[\frac{1}{w}p_I A'_{II} + L'_{II}]b + [p_I(A'_{II} - A_{II}) + w(L'_{II} - L_{II})]b = 0 \tag{35}$$

which becomes

$$\frac{w' - w}{w} = \frac{[w(L_{II} - L'_{II}) - p_I(A'_{II} - A_{II})]b}{[p_I A'_{II} + wL'_{II}]b} \tag{36}$$

Step 2. Now assume the rate of exploitation remains constant before and after a CU-LS technical change in Dept-Con. Then both the new real wage index and the new rate of profit will presumably be different from the old ones. Denote the new levels as w^* and π^*. By the constancy of the rate of exploitation,

$$e = \frac{1 - \Lambda_{II}wb}{\Lambda_{II}wb} = \frac{1}{\Lambda_{II}wb} - 1 = \frac{1}{\Lambda'_{II}w^*b} - 1 \tag{37}$$

So the value of labor power will remain constant, $\Lambda_{II}wb = \Lambda'_{II}w^*b$, with which we have

$$\frac{w^* - w}{w} = \frac{w^*}{w} - 1 = \frac{\Lambda_{II}b - \Lambda'_{II}b}{\Lambda'_{II}b} \tag{38}$$

$$= \frac{[\Lambda_I(A_{II} - A'_{II}) + L_{II} - L'_{II}]b}{[\Lambda_I A'_{II} + L'_{II}]b} \tag{39}$$

Step 3. Now compare the two wage differences in Equation (36) and (39), it's obvious that the difference between w' and w^* lies in the difference between p_I and $w\Lambda_I$. Since

$$p_I = wL_I[1/(1+\pi)I - A_I]^{-1} \tag{40}$$

$$w\Lambda_I = wL_1(I - A_I)^{-1} \tag{41}$$

we only need to compare $[1/(1+\pi)I - A_I]^{-1}$ and $[I - A_I]^{-1}$.

Because

$$M \equiv A + bL \geq A \geq \bar{A}_I \equiv \begin{pmatrix} A_I & 0 \\ 0 & 0 \end{pmatrix} \tag{42}$$

by the Perron-Frobenius theorems, we have the following relationships regarding the largest eigenvalues of these matrices,

$$1 > 1/(1+\pi) = \lambda(M) > \lambda(A) \geq \lambda(\bar{A}_I) = \lambda(A_I) \tag{43}$$

The first and inequality signs are strict because M is reproducible and indecomposable; the third inequality sign is strict because both A and $\bar{A}$ are decomposable. The first equality sign is because of the price system; the second equality sign is because the eigen-systems of $\bar{A}_I$ and A_I are equivalent. Thus $[1/(1+\pi)I - A_I]^{-1} \geq 0$, $[I - A_I]^{-1} \geq 0$ and

$$1/(1+\pi)I - A_I \leq I - A_I \tag{44}$$

Post-multiply the above inequality by $[1/(1+\pi)I - A_I]^{-1}$, then post-multiply by $[I - A_I]^{-1}$, we have

$$[1/(1+\pi)I - A_I]^{-1} \geq [I - A_I]^{-1} \tag{45}$$

which implies $p_I > w\Lambda_I$ from Equation (40) and (41). Because of this, we have $w^* > w'$, from Equation (36) and (39). At last, because the wage-profit frontier is always downward sloping, we have $\pi^* < \pi$. □

C Further generalization considering capital stock

Capital stock is a better name for fixed capital (as is often used in the literature) in the sense that capital tied up in actual productions also includes inventories of raw materials that are not fixed like machinery, but rather necessary for smooth production. With the presence of capital stock, we are going to make two more assumptions: First we assume the turn over times for each type of intermediate inputs remain the same after a technical change, so as to control for the turnover time effect that Franke (1999) fails to do. Second, for simplicity, we assume wage is post-paid such that the rate of profit is only the markup over capital stock.

Let B be the $m \times m$ capital stock matrix whose generic element b_{ij} means the amount of capital stock of good i needed to produced one unit of good j. Let T be the $m \times m$ turnover time matrix whose generic element t_{ij} means the turnover time of capital stock of good i used in producing good j. Furthermore, $B = A \odot T$ where $\odot$ means element by element multiplication, i.e., $b_{ij} = a_{ij} \times t_{ij}$. (Brody 1970: 35-41)

We need to first redefine the two types of technical changes that we are interested in since capital stock is being considered (this case is distinguished by using the super subscript Ⓢ).

Definition 3. *A technical change in sector j, $(B_j, A_j, l_j) \mapsto (B'_j, A'_j, l'_j)$, is capital-using and labor-saving (CU-LS*Ⓢ*) if and only if the production process uses more intermediate inputs (both fixed can circulating) evaluated at current prices, and less direct labor input, i.e.,*

$$pB'_j > pB_j \quad and \quad pA'_j > pA_j \quad and \quad l'_j < l_j \tag{46}$$

Definition 4. *A technical change in sector j, $(B_j, A_j, l_j) \mapsto (B'_j, A'_j, l'_j)$, is cost-reducing*Ⓢ *if and only if it can generate a higher temporary rate of profit under current price levels for the sector where it takes place, i.e.,*

$$(\pi + \Delta\pi)pB'_j + pA'_j + pw\bar{b}l' = p_j \quad and \quad \Delta\pi > 0 \tag{47}$$

Under these two definitions, it could be similar proved that there will always be an overlapping between CU-LSⓈ and cost-reducingⓈ technical changes in a "cost-space" similar to the case without capital stock.

Theorem 3. *Assuming the rate of exploitation constant, for a CU-LS*Ⓢ *technical change,*

(i) If it takes place in Dept-Con, then the price rate of profit will fall.

(ii) If it takes place in Dept-Cap, then the direction of movement in the rate of profit will be indeterminate.

Proof. (i) First let the real wage index w adjust so as to maintain constancy of the rate of profit after a CU-LSⓈ technical change in Dept-Con. The price system before the technical change is

$$p_I = \pi p_I B_I + p_I A_I + p_{II} w b L_I \tag{48}$$

$$p_{II} = \pi p_I B_{II} + p_I A_{II} + p_{II} w b L_{II} \tag{49}$$

$$p_{II} b = 1 \tag{50}$$

So $p_I = w L_I P^{-1}$ where $P \equiv I - \pi B_I - A_I$. The price system after the technical change becomes

$$p'_I = \pi p'_I B_I + p'_I A_I + p'_{II} w' b L_I \quad or \quad p'_I = w' L_I P^{-1} \tag{51}$$

$$p'_{II} = \pi p'_I B'_{II} + p'_I A'_{II} + p'_{II} w' b L'_{II} \tag{52}$$

$$p'_{II} b = 1 \tag{53}$$

Changes in the prices are

$$p'_I - p_I = (w' - w) L_I P^{-1} = \frac{w' - w}{w} p_I \tag{54}$$

$$p'_{II} - p_{II} = \pi p'_I B'_{II} - \pi p_I B_{II} + p'_I A'_{II} - p_I A_{II} + w' L'_{II} - w L_{II} \tag{55}$$

$$= \pi(p'_I - p_I) B'_{II} + \pi p_I (B'_{II} - B_{II}) + (p'_I - p_I) A'_{II} + p_I (A'_{II} \tag{56}$$

$$- A_{II}) + (w' - w) L'_{II} + w (L'_{II} - L_{II}) \tag{57}$$

Post multiply the price differences of Department II by b and apply the numeraire equations, we have

$$
\begin{aligned}
0 &= (p'_{II} - p_{II})b \\
&= [\pi(p'_I - p_I)B'_{II} + \pi p_I(B'_{II} - B_{II}) + (p'_I - p_I)A'_{II} + p_I(A'_{II} \\
&\quad - A_{II}) + (w' - w)L'_{II} + w(L'_{II} - L_{II})]b
\end{aligned} \tag{58}
$$

Use the price changes in Department I and do some rearrangement, we arrive at

$$
\hat{w}' = \frac{w' - w}{w} = \frac{[-\pi p_I(B'_{II} - B_{II}) + w(L_{II} - L'_{II}) - p_I(A'_{II} - A_{II})]b}{[\pi p_I B'_{II} + p_I A'_{II} + wL'_{II}]b} \tag{59}
$$

Next, derive exactly the same $\hat{w}^*$ since the value system is by definition not related to the capital stock at all.

$$
\frac{w^* - w}{w} = \frac{[\Lambda_I(A_{II} - A'_{II}) + L_{II} - L'_{II}]b}{[\Lambda_I A'_{II} + L'_{II}]b} \tag{60}
$$

Now let

$$
\hat{w}^0 \equiv \frac{[w(L_{II} - L'_{II}) - p_I(A'_{II} - A_{II})]b}{(p_I A'_{II} + wL'_{II})b} \tag{61}
$$

then we have $\hat{w}^0 > \hat{w}'$ since we assume π to be positive, and by CU-LS$^{Ⓢ}$, $p_I B'_{II} \geq p_I B_{II}$. Notice the difference between $\hat{w}^0$ and $\hat{w}^*$ lies in the difference of $p_I/w = L_I(I - \pi B_I - A_I)^{-1}$ and $\Lambda_I = L_I(I - A_I)^{-1}$. Then $p_I/w > \Lambda$ because, again, $\pi > 0$. Therefore $\hat{w}^* > \hat{w}^0 > \hat{w}'$. By the downward sloping wage-profit frontier, $\pi^* < \pi$. Note that the crucial condition in this proof is $p_I B'_{II} \geq p_I B_{II}$, which allows for any change in the turnover times of the intermediate inputs, meaning the theorem also holds under constant turnover times, which overcomes the shortfall of Franke (1999)'s model whose result is partly driven by increasing turnover times.

(ii) This part holds because of the validity of Laibman (1982)'s two-sector model with post-paid wage, and the fact that the case without capital stock is a special case of the general case with capital stock. □

References

Abacus (2019). Chinese browsers block protest against china's 996 overtime work culture. `https://www.abacusnews.com/digital-life/chinese-browsers-block-protest-against-chinas-996-overtime-work-culture/article/3004543`. Accessed: April 6, 2019.

Basu, D. (2018). Reproduction and crisis in capitalist economies. In *The Oxford Handbook of Karl Marx*.

Bidard, C. (2004). *Prices, reproduction, scarcity*. Cambridge University Press.

Bowles, S. and H. Gintis (1978). Professor morishima on heterogeneous labour and marxian value theory. *Cambridge Journal of Economics 2*(3), 311–314.

Braverman, H. (1974). Labor and monopoly capital. *New York: Monthly Review*.

Brody, A. (1970). *Proportions, prices and planning*. Akademiai Kiado.

Carchedi, G. and M. Roberts (2013). A critique of heinrich's,'crisis theory, the law of the tendency of the profit rate to fall, and marx's studies in the 1870s'. *Monthly Review website, December 1*, 2013.

Clawson, D. (1980). *Bureaucracy and the labor process: The transformation of US industry, 1860-1920*. NYU Press.

Flaschel, P., R. Franke, and R. Veneziani (2013). Labour productivity and the law of decreasing labour content. *Cambridge Journal of Economics 37*(2), 379–402.

Foley, D. K. (2009). *Understanding capital: Marx's economic theory*. Harvard University Press.

Franke, R. (1999). Technical change and a falling wage share if profits are maintained. *Metroeconomica 50*(1), 35–53.

Github (2019). Project 996.icu. https://github.com/996icu/996.ICU. Accessed: April 6, 2019.

Guardian (2019a). Karl marx's london grave vandalised in suspected hammer attack. https://www.theguardian.com/books/2019/feb/05/karl-marx-london-grave-vandalised-suspected-hammer-attack-highgate-cemetery. Accessed: February 17, 2019.

Guardian (2019b). Karl marx's london memorial vandalised for second time. https://www.theguardian.com/uk-news/2019/feb/16/karl-marxs-london-memorial-vandalised-for-second-time. Accessed: February 17, 2019.

Han, Z. and B. Schefold (2006). An empirical investigation of paradoxes: reswitching and reverse capital deepening in capital theory. *Cambridge Journal of Economics 30*(5), 737–765.

Heinrich, M. (2013a). Crisis theory, the law of the tendency of the profit rate to fall, and marx's studies in the 1870s. *Monthly Review 64*(11), 15.

Heinrich, M. (2013b). Heinrich answers critics. *Monthly Review*.

Keynes, J. M. (2010). Economic possibilities for our grandchildren. In *Essays in persuasion*, pp. 321–332. Springer.

Kliman, A., A. Freeman, N. Potts, A. Gusev, and B. Cooney (2013). The unmaking of marx's capital: Heinrich's attempt to eliminate marx's crisis theory.

Kurz, H. D. (2018). Will the mega2 edition be a watershed in interpreting marx? *The European Journal of the History of Economic Thought*, 1–25.

Laibman, D. (1982). Technical change the real wage and the rate of exploitation: The falling rate of profit reconsidered. *Review of Radical Political Economics 14*(2), 95–105.

Laibman, D. (2018). Money and totality: Another round of debate on value formation and transformation. *World Review of Political Economy 9*(1), 21–40.

Mage, S. (2013). Response to heinrich—in defense of marx's law. *Monthly Review 65*(4).

Mantoux, P. (1961). *The industrial revolution in the eighteenth century: An outline of the beginnings of the modern factory system in England*. Harper & Row.

Marx, K. (1962). *Capital: a critique of political economy, Volume III*. Foreign Languages Publishing House.

Mohun, S. (2005). On measuring the wealth of nations: the us economy, 1964–2001. *Cambridge Journal of Economics 29*(5), 799–815.

Morishima, M. (1973). *Marx's Economics: A Dual Theory of Value and Growth*. Cambridge University Press.

Moseley, F. (2013). Critique of heinrich: Marx did not abandon the logical structure. *Monthly Review 1*.

Nakatani, T. (1980). The law of falling rate of profit and the competitive battle: comment on shaikh. *Cambridge Journal of Economics 4*(1), 65–68.

Nakatani, T. and T. Hagiwara (1978). The rate of profit, real wages and technical change: Considering durable equipment. *The Economic Review 29*(1). (in Japanese).

Nakatani, T. and T. Hagiwara (1997). Product innovation and the rate of profit. *Kobe University economic review 43*, 39–51.

Nikaido, H. (1970). *Introduction to sets and mappings in modern economics*. North-Holland/American Elsevier.

Noble, D. (2017). *Forces of production: A social history of industrial automation*. Routledge.

Noble, D. F. (1979). Social choice in machine design: The case of automatically controlled machine tools. *Case studies on the labor process*, 18–50.

Okishio, N. (1961). Technical changes and the rate of profit. *Kobe University Economic Review 7*, 85–99.

Okishio, N. (2001). Competition and production prices. *Cambridge Journal of Economics 25*(4), 493–501.

Pun, N. and J. Chan (2012). Global capital, the state, and chinese workers: the foxconn experience. *Modern China 38*(4), 383–410.

Roemer, J. E. (1978). The effect of technological change on the real wage and marx's falling rate of profit. *Australian Economic Papers 17*(30), 152–166.

Roemer, J. E. (1979). Continuing controversy on the falling rate of profit: fixed capital and other issues. *Cambridge Journal of Economics 3*(4), 379–398.

Roemer, J. E. (1981). *Analytical foundations of Marxian economic theory.* Cambridge University Press.

Saad-Filho, A. (1993). A note on marx's analysis of the composition of capital. *Capital & Class 17*(2), 127–146.

Shaikh, A. (1978). Political economy and capitalism: notes on dobb's theory of crisis. *Cambridge Journal of Economics 2*(2), 233–251.

Shaikh, A. (1987). The poverty of algebra. *Alternatives to economic orthodoxy: A reader in political economy*, 297–322.

Shaikh, A. (1990). Organic composition of capital. In *Marxian Economics*, pp. 304–309. Springer.

Shaikh, A. (2016). *Capitalism: Competition, conflict, crises.* Oxford University Press.

Sward, K. (1968). *The Legend of Henry Ford*, Volume 129. New York: Atheneum, 1968 [c1948].

Van Parijs, P. (1980). The falling-rate-of profit theory of crisis: A rational reconstruction by way of obituary. *Review of Radical Political Economics 12*(1).

Vice (2019). Chinese workers are trying to bake fair labor practices into software. `https://motherboard.vice.com/en_us/article/mbz84n/chinese-workers-are-trying-to-bake-fair-labor-practices-into-software`. Accessed: April 6, 2019.